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EXPOSITORY REMARKS ON THE THEORY OF GROUPS.*

BY G. A. MILLER,
University of Illinois.

OFFI... 2... MA...

The technical term *group* appears in the mathematical literature with widely different meanings. For instance, the explicit definitions of this term which appear in the *Encyclopédie des Sciences Mathématiques*, tome 1, volume 1, page 576, and tome 1, volume 2, page 243, respectively, differ widely. The latter of these definitions is too general, to serve as a basis for any extensive theory in the present state of mathematical development. In fact, the former is also too general, if it is assumed that it is supposed to apply to groups of infinite order as seems to be implied by the first sentence which follows this definition At any rate, it is the definition of a semi-group, *cf.* Pascal's *Repertorium der höheren Mathematik*, volume 1, (1910), page 191. The same unsatisfactory definition appears in the German edition of the said Encyclopaedia, Volume 1, page 217, where it is stated that it applies also to groups of infinite order, and hence the student of mathematics cannot find satisfactory information relating to the definition of the term group by consulting our best modern mathematical encyclopaedias.

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* [*The Mathematics Student* hopes to publish, from time to time, expository papers dealing with developments in important topics in Modern Mathematics. A paper by Dr. Chowla reviewing the progress in the Theory of Numbers appeared in Vol. I. pages 41—48. The Editor wishes to thank Prof. Miller for having kindly responded to a similar request for a review article on Group Theory. —ED.]

One of the simplest definitions of the technical term *group*, which applies to the groups of infinite order as well as to those of finite order may be stated as follows: A set of distinct elements $s_1, s_2, s_3, \ldots \ldots$ constitutes a group if it satisfies the conditions that if any two of the three symbols in the equation $xy = z$ are replaced by the same element or by two different elements of this set, the equation will be satisfied by one and only element of the set. Moreover, the elements obey the associative law when they are combined successively two at a time or one with its powers. This definition appears in the treatise on *Finite Groups* by Miller, Blichfeldt, Dickson, 1916, page 52. An equivalent definition is found in H. Weber's *Kleines Lehrbuch der Algebra*, 1912, page 181, but a redundant condition is given here as the first condition and hence Weber's definition is unnecessarily long and therefore undesirable.

The fact that only two elements of a group are supposed to be combined at one time has not been sufficiently emphasized in the definitions of the technical term *group*. To see that this emphasis is desirable, it is only necessary to note that if in a given group whose order exceeds 2 we multiply a product of two or more elements by the number of these elements, and this number is not constant, the result is not a group. Even the common positive rational numbers when they are combined by multiplication do not constitute a group if the products, consisting of different numbers of factors, are always multiplied by the number of these factors. It should be noted that this method of combining the rational numbers is a perfectly definite operation but it is not a group operation. It is desirable to distinguish clearly in the mathematical developments between group operations and non-group operations. *Cf.* G. A. Miller, *Proceedings of the National Academy of Sciences of the United States of America*, Volume 18 (1932), page 598.

Another feature of the definition of a group, which seems to deserve more emphasis than it has received in the published definitions of this term, is the meaning of the phrases "any two," "every two," or "every

pair." In the former of the two definitions to which we referred near the beginning of this article, the first of these phrases is used. The exact meaning of these phrases is important in group theory but it is not always correctly understood by the beginner. If there are n distinct objects, the number of the possible pairs is $n(n-1)/2$ or $n(n+1)/2$ according as no repetitions or as repetitions are considered. The former represents the number of simple pairs while the latter represents the number of complete pairs. In elementary mathematics the student often becomes familiar only with the former types of pairs, but in the definitions of the technical term *group*, the latter are commonly implied even when this is not explicitly stated. In the volume noted at the close of the preceding paragraph, page 100, all the groups are determined in which it is possible to select a set of generating operators in such a way that they include every simple pair thereof but not every complete pair, and hence the set does not constitute a group. This exhibits the great desirability of an explicit statement relating to this point in the definitions of a group.

Many of the difficulties encountered by the student of mathematics are inherent in the subject and these can be overcome only by serious meditations; but many others are due to a lack of clearness in the presentations, and these should be banished from mathematical literature as rapidly as possible. The literature of group theory is still in need of many such banishments and the present article aims to be effective in this direction. The equational definition of the technical term *group* which is found in the second paragraph of this article should replace many of the longer definitions of this term. According to this definition, the natural numbers do not constitute a group when they are combined by multiplication as may be seen by substituting in the given equation a prime number for z and a different prime number for x. Notwithstanding this obvious contradiction, one of the most widespread errors relating to our subject is the statement that the natural numbers constitute a group when they are combined by multiplication. In the first edition of Volume 2 of H. Weber's *Lehrbuch der Algebra*, 1896,

page 54, it is stated that this even constitutes the most important example of an abelian group, but this statement was fortunately corrected in the second edition of this volume.

Even the cyclic group of infinite order has been treated quite differently by different writers. It is sometimes said that this group has two independent generators, s_1, s_2 which satisfy the equation $s_1s_2 = 1$. *Cf.* Pascal's *Repertorium der höheren Mathematik,* Volume 1 (1910), page 193. If this is done, it results that different powers of the same operator are independent operators. In particular, if we consider the cyclic group of infinite order composed of the powers of 2, it would result that 2 and $\frac{1}{2}$ are its independent generators even though both of these numbers are powers of 2. While there are some advantages in this point of view, it must be regarded as a convention which establishes an unnecessarily wide difference between the treatment of the cyclic groups of finite and of infinite order. It would seem wiser to say that a cyclic group of infinite order as well as a cyclic group of finite order has a single generator and that each of its operators is a power of this generator, as is done, for instance, by H. Weyl, *Gruppentheorie und Quantenmechanik,* 1931, page 105.

The definitions of the abstract groups of finite order were usually based on the properties of permutation groups since it was found early in the development of group theory that every such abstract group can be represented as a permutation group. On the contrary, there has not been such a useful guiding principle in the formulation of the definitions of abstract groups of infinite order and hence a greater lack of uniformity may be observed in the latter definitions than in the former. Even such an eminent authority on groups of an infinite order as S. Lie asserted that there are groups which contain neither the identity nor the inverses of their operators; *Theorie der Transformationsgruppen,* Volume 1 (1888), page 163. Such assertions merely imply that S. Lie included here under the term group what other writers call semi-groups. Although such variations in the use of the same term impose unnecessary difficulties on the beginner, they can scarcely be avoided in the developments of a subject in which a large number take an active part.

Suppose that G represents a finite group of order g and that s represents any one of its operators except the identity. The successive powers of s are also operators of G and hence not more than g of them can be distinct. If s^m is the first power of s which is equal to a lower power thereof, then s^{m-1} is the identity of G. If m is less than g, then G contains a proper sub-group of order m. If m is equal to g but composite, then a power of s generates a proper sub-group of G. Hence it results that every group, except the groups of prime orders, contains a proper sub-group and hence its operators can be arranged in the form of a rectangle in which this proper sub-group is the first row. Such an arrangement appears in a letter from P. Abbati to P. Ruffine, dated September 30, 1802, and hence it is one of the earliest developments in the theory of groups and also one of the most important since it is useful not only in establishing the fact that the order of a group is divisible by the order of every one of its sub-groups, but also to prove that the operators of every other sub-group of G are equally distributed among the rows which involve at least one such operator.

From the preceding paragraph, it results that it was easy to prove that every group whose order is not a prime number, contains at least one proper sub-group. It was much more difficult to find, in general, the order of such a sub-group. The first step in this direction was taken by E. Galois, who asserted without giving any proof, as far as we know, that whenever g is divisible by a prime number p, then G contains a sub-group of order p. The earliest known proof of this fundamental fact appears in A. L. Cauchy's *Exercices d' Analyse,* Volume 3, 1844. L. Sylow later (1872) embodied it in what is now commonly known as Sylow's theorem, which establishes the fact that if p^m is the highest power of p which divides g, then G contains at least one sub-group of order p^m. If it contains more than one such sub-group, the number of these sub-groups is of the form $1 + kp$, and they are all conjugate under G. This theorem establishes the existence of some but not necessarily all the sub-groups which appear in a group whose order is given.

Sylow's theorem gives some information in regard to the number of the sub-groups of order p^m which can appear, but it gives very little

information in regard to the number of the operators whose orders are powers of p which appear in G. It is easy to prove that every such operator appears in at least one of the Sylow sub-groups of G and that G contains at least p^{m+1} operators whose orders are powers of p whenever it contains more than one Sylow sub-group or order p^m; *Cf.* Miller, Blichfeldt, Dickson, *Finite Groups*, 1916, page 80. No one knows how many theorems on group theory are now in the mathematical literature. In the second edition (1927) of his excellent *Theorie der Gruppen von endlicher Ordnung*, A. Speiser numbers his theorems from 1 to 217; but he naturally does not include nearly all of the known theorems on this subject, and various theorems can often be combined into a more general one. It would not be difficult to make a list of more than 500 theorems relating to groups of finite order alone, and there is still room for many new and interesting ones.

In support of this last assertion, we proceed to develop here a few new theorems relating to an elementary question in group theory, *viz.*, the number of operators whose orders are a power of a prime number p in a group whose order is divisible by p^m but by no higher power of p. One of the fundamental questions in this connection is whether there are groups which have a set of Sylow sub-groups such that none of these sub-groups involves an operator which does not also appear in another such sub-group. To prove a theorem according to which this question can be answered in the affirmative, it is desirable first to note the theorem that the number of the Sylow sub-groups of order 2^m in any given symmetric group of degree n is $n!/2^m$. This theorem results directly from the fact that when n is of the form 2^k, then the Sylow sub-group of order 2^m contained in the symmetric group of degree n is the direct product of two Sylow sub-groups contained in the symmetric group of degree $n/2$ extended by an operator of order 2 which merely interchanges these two Sylow sub-groups. Hence such a Sylow sub-group contains one and only one substitution of order 2 which is commutative with every one of its substitutions.

From the method used in the preceding paragraph to construct a

Sylow sub-group of order 2^m contained in the symmetric group of degree $2k$, it is easy to see how a Sylow sub-group of order 2^{m-1} of the corresponding alternating group can be constructed and that when $k > 2$ there is one and only one substitution of order 2 which is commutative with every substitution of this alternating group and contained therein. Such a substitution of order 2 appears in more than one of the Sylow sub-groups of order 2^{m-1} contained in this alternating group ; but if two Sylow sub-groups of this order in this alternating group involve different invariant substitutions of order 2 they are necessarily distinct. Greater details in regard to these developments appear in an article by the present writer and published under the title "On the transitive substitution groups whose order is a power of a prime number," *American Journal of Mathematics*, Volume 23 (1901), page 173.

It is now easy to prove that when $k > 3$ is even, the Sylow subgroups whose orders are a power of 2 in the alternating group of degree 2^k have the property that every substitution contained in one of them appears also in at least one of the others. To prove this theorem it is only necessary to observe that every such substitution is commutative with more than one substitution of order 2 and of degree 2^k which appears in this alternating group. If a positive substitution whose degree does not exceed 2^k is commutative with only one substitution of order $2k$ on these 2^k letters, all its cycles have different orders and hence they omit at least four letters. As different substitutions of order 2 can be formed on the omitted letters, the theorem in question has been proved. With the increase of the publications on group theory, the beginner in this field is compelled to familiarize himself with a larger number of theorems relating thereto before he can be sure that his discoveries are new ; but he also has the advantage that the border line between the known and the unknown is continually getting longer, and hence there is a larger variety of problems which await solution.

The development of group theory during the nineteenth century gained momentum as the century advanced and represents one of the

most prominent features of the mathematical developments of this century. In reviewing these developments, J. Pierpont made the following remark: "In resume, we may thus say that the group concept, hardly noticeable at the beginning of the century, has at its close become one of the fundamental and most fruitful notions in the whole range of our science." *Bulletin of the American Mathematical Society*, Volume 11 (1904), page 144. At the beginning of the century it was regarded merely as a useful concept in the development of a general theory of algebraic equations while at the end thereof it had permeated so many mathematical theories that H. Poincare was led to remark as follows: "The theory of groups is so to say entire mathematics divested of its matter and reduced to pure form," *Acta Mathematica*, Volume 38 (1921), page 145. A recent very useful brief summary of the most fundamental developments in this field may be found in an Italian work entitled *Enciclopedia delle Matematiche Elementari*, Volume 1, part 2 (1932), pages 19—69. This extensive article was written by L. Berzolari and is especially interesting because it appears in a work on elementary mathematics and consequently it emphasizes the fundamental ideas of our subject.

GLEANINGS.

24. We reject the thesis of the categorical finiteness of man, both in the atheistic form of obdurate finiteness which is so alluringly represented to-day in Germany by the Freiburg philosopher Heidegger, and in the theistic, specifically Lutheran-Protestant form, where it serves as a back-ground for the violent drama of contrition, revelation and grace. On the contrary, mind is freedom within the limitations of existence; it is open towards the infinite. Indeed, God as the completed infinite cannot and will not be comprehended by it; neither can God penetrate into man by revelation, nor man penetrate to him by mystical perception. The completed infinite we can only represent in symbols. From this relationship every creative act of man receives its deep conservation and dignity. But only in mathematics and physics, as far as I can see, has symbolical-theoretical construction acquired sufficient solidity to be convincing for every one whose minds open to these sciences.

DR. HERMAN WEYL: *The open World*.

25. MEMORISING LOG π

The logarithm of *pi* to the base ten is often used in sceintific problems. To memorize this value to 20 decimal places, memorize the following rhyme Each word represents the digit of the number which is determined by the number of letters in the word. "No" represents Zero; $\log_{10} \pi = 0.49714\ 987\ 2694133\ 85435$.

No task, therefore, becomes a bore
Following pleasant Science
It simply increases what I use, and
Supplies might, like the lion's,.

Earl C. Rex. M. S. in *The Scientific American*, May 1934.

SOME RESULTS CONNECTED WITH TRIANGLES IN PERSPECTIVE.*

By K. Satyanarayana,
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In what follows, O is the centre and XYZ the axis of perspective of two triangles ABC, A′B′C′; and A″, B″, C″ are points on OA, OB, OC such that OA″ = A′A, etc.; (F), (F′), (F″) are the circum-conics of OABC, OA′B′C′, OA″B″C″; and Γ is the conic for which the triangles ABC, A′B′C′, are reciprocal.

It is proposed to present in this paper two types of results. The first type developed in paras 1 to 4 is in respect of certain lines (designated *polar lines* and *polar axes*) connected with the triangles and their relations to Γ and to the *s. s. p.*† members Σ, Σ′, Σ″ of (F), (F′), (F″); the condition that each of two triangles in perspective shall be self-conjugate for the same conic; and the condition that Γ be a parabola. The second type outlined in the rest of the paper relates to the theory of the rigid relative displacement of the triangles without disburbance of perspectivity.

A detailed discussion of the results given in Question 1648 by Mr. Ramaswamy Aiyar will be found in the last para.

1. On certain lines connected with the triangles.

Let a transversal cut OA, OB, OC in L, M, N; and let the intersections of YN, ZM; ZL, XN; XM, YL be respectively H_1, H_2, H_3. By projecting XYZ to infinity and O into the common ortho-centre of *abc* and *a′b′c′*‡, it is easily seen that h_1, h_2, h_3 are respectively the ortho-centres of *omn*, *onl*, *olm* and lie on the perpendicular from *o* on *lmn*. Obviously to all lines parallel to *lmn* corresponds the same line $h_1h_2h_3$. Also the parallel to *lmn* through *o* and the line $h_1h_2h_3$ are conjugate rays of the involution pencil *oa*, *ox*; *ob*, *oy*; *oc*, *oz*. Hence

To *every* ray LMN of a pencil with its vertex V on the axis of perspective corresponds the same line $OH_1H_2H_3$. We shall call this line the *polar line* ‖ *of perspective* of V. ... (1.1)

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* I am indebted to Mr. V. Ramaswamy Aiyar whose Questions 1648, 1650, 1651, 1660, *Math. Stud.*, Vol. I have helped in this investigation.

† '*s. s. p.*' is used as an abbreviation for 'similar and similarly placed.'

‡ Small letters indicate the projections.

‖ For the significance of this word, *vide* result (4.11).

OV and the polar line of V are conjugate rays of the involution pencil of perspective, *viz.*, OA, OX ; OB, OY ; OC, OZ. ... (1.2)

The polar line of the point at infinity on XYZ may in particular be called the *polar axis* of perspective of the triangles. It corresponds to all lines LMN parallel to XYZ as well as to the line at infinity, and cuts XYZ in the centre of the involution range formed on XYZ by the involution pencil of perspective. ... (1.31)

By considering the polar axis as corresponding to the line at infinity, it is easy to see that :—

If O is the ortho-centre of ABC, then the polar axis contains the ortho-centres of the triangles AYZ, BZX, CXY. ... (1.32)

2. Certain lemmas on *s. s. p.* conics.

The following lemmas on *s. s p.* conics may be proved directly or by projecting them into circles. We take Σ, Σ' to be two *s. s. p.* conics cutting in O, O′.

If AOA′, PO′P′ be two chords, AP is parallel to A′P′ ; conversely if AOA′ be a chord and AP, A′P′ be parallel, then PO′P′ is a straight line. ... (2.1)

OAA′, OBB′ being any two chords, if AB is parallel to A′B′, then Σ, Σ' touch at O. ... (2.2)

If Σ, Σ' touch at O and OAA′, OBB′, OCC′ be any three lines cutting Σ, Σ' respectively in A, B, C ; A′, B′, C′, then the triangles ABC, A′B′C′ are *s. s. p.*, O being the centre of similitude. ... (2.3)

If on *any* chord OAA′, a point P be taken such that OP = AA′, the locus P is a *s. s. p.* conic through O touching the common chord OO′. ... (2.4)

Its semi-diameter through O equals the distance between the centres of Σ and Σ'.

3. *s. s. p.* members of (F), (F′), (F″).

If the triangles ABC, A′B′C′ be *s s p*, then to any conic of (F) corresponds a unique conic of (F′) *s. s. p*, to the former and the two touch each other. ... ... (3.1)

For taking any member Σ of (F), consider the *s. s. p.* conic through O, A′, B′. This touches Σ by (2.2) and passes through C by (2.3).

If the triangles ABC, A′B′C′ are not similarly placed, there exists (except when O is at infinity) a unique member Σ of (F) and a unique member Σ' of (F′) which are *s. s. p.* ... (3.2)

For, the pencils of conics through OABC and OA′B′C′ cut the line at infinity in pairs belonging to two involutions, and we know there is always one pair common to two involutions. The conics through these points fulfil the required conditions. If O is at infinity, it is easy to show that to every member of (F) corresponds a *s. s. p.* member of (F′).

It therefore follows that, when O is not at infinity and the triangles ABC, A′B′C′ are not similarly placed, if two *s. s. p.* members of (F), (F′) be discovered, these should be Σ, Σ'. For example:

If OABC, OA′B′C′ be cyclic quadrilaterals, Σ, Σ' are the circum-circles: ... (3.21)

If a single pair of corresponding sides, say AB and A′B′, be parallel, then Σ, Σ' are respectively AB, OC; A′B′, OC′. ... (3.22)

If A, B, C be collinear, Σ consists of ABC and another line through O. So, if A′, B′, C′ be not collinear but a side, say A′B′, is parallel to ABC, then Σ, Σ' consist respectively of ABC, OC and A′B′, OC ... (3.231)

If A′, B′, C′ be also collinear (but not parallel to ABC), Σ consists of ABC and a parallel through O to A′B′C′ and Σ' consists of A′B′C′ and a parallel through O to ABC. ... (3.232)

From (2.4) it follows that when Σ, Σ' are the conics under reference in that result, a conic Σ'' *s. s. p.* to Σ, Σ' belongs to (F″) Hence from (3.2), it follows that :—

If the triangles ABC, A′B′C′ be not similarly placed, then (*F*), (*F′*), (*F″*) *possess respectively three unique s. s. p. conics* Σ, Σ', Σ''. ... (3.3)

It may be mentioned that the triangle A″B″C″ cannot be similarly placed to either of the triangles ABC, A′B′C′.

The polar axis of the triangles ABC, A′B′C′ is the common chord of Σ, Σ' *and touches* Σ''. ... (3.4)

To prove this, project O into the common ortho-centre of the triangles abc, $a'b'c'$. Then the projections of Σ, Σ' are rectangular hyperbolas having the projection of the line at infinity as a common chord. Hence, since the common points of two rectangular hyperbolas form an ortho-centric tetrad, the other common chord through the projection of O must be perpendicular to the projection of the line at infinity. So in the original figure the common chord is the polar axis. From (2.4) it follows that this polar axis touches Σ''. Hence

If Σ, Σ' be two *s. s, p.* conics through O and if OAA′, OBB′, OCC′ be *any* three chords, the triangles ABC, A′B C′ have always the same polar axis, namely, the common chord of Σ, Σ'. ... (3.5)

4. The Conic Γ.

Since the involution pencil of perspective is the pencil of conjugate lines of Γ, it follows that

V and its polar line of perspective are pole and polar for Γ and the polar axis of perspective is the diameter of Γ conjugate to the axis of perspective. ... (4.11)

If the involution pencil of perspective be orthogonal, O and XYZ being then a focus and the corresponding directrix, the polar axis is a principal axis of Γ. ... (4.12)

The Γ's of all pairs of triangles in perspective with their corresponding vertices on three fixed straight lines and with their corresponding sides concurrent at three fixed points have double contact. ... (4.2)

This follows since the involution of perspective in each case is the same. The double contact is at the double points of the involution range formed on XYZ.

Let XYZ cut OA, OB, OC in α, β, γ. If A_1, A'_1 belong to the involution (Oα, AA′), ZA , ZA_1' cut OB in B_1, B_1', belonging to the involution (Oβ, BB′) and YA_1, YA'_1 cut OC in C_1, C'_1 belonging to the involution (Oγ, CC′). Project O into the common ortho-centre of the triangles abc, $a'b'c'$. Then o is the ortho-centre of the triangles $a_1b_1c_1$, $a'_1b'_1c'_1$. Also o is the centre of the projections of the involution ranges so that $oa.oa' = oa_1.oa'_1$. Hence $a_1b_1c_1$, $a_1'b_1'c_1'$ are reciprocal for the circle for which abc, $a'b'c'$ are reciprocal. Hence pairs of triangles like $A_1B_1C_1$, $A'_1B'_1C'_1$ are also reciprocal for Γ. ... (4.31)

It is easy to prove that, if L, L′ ; M, M′ ; N, N′ be the respective double points of the involutions on OA, OB, OC, there are two triangles LMN, L′M′N′ each self-conjugate for Γ. ... (4.32)

The solution of the proper vertices of these triangles is to be made as in (4.31). From the above, incidentally it follows that :

The necessary and sufficient condition that two triangles in perspective shall be each self-conjugate for the same conic is that each pair of corresponding vertices should be separated harmonically by the centre and axis of perspective. ... (4.33)

This can be proved by projection as in (4.31).

If Σ, Σ' cut in O, O′, the centre of Γ is O′. ... (4.4)

To prove this, let O′A cut Σ'' in P. Then by (2.1) PB′, PC′ are respectively parallel to AB, AC. The polars of P, B′, C′ with respect to Γ form a triangle in perspective with the tiangles PB′C′. But PB′ and AB (the polar of C′) are parallel, and likewise, PC′ and AC (the polar of B′). Hence B′C′ must be parallel to the polar of P. So O′AP is a diameter of Γ since the polars of A and P are parallel. Hence the result (3.5) now takes the form :—

If Σ, Σ' be two *s. s, p.* conics through O and if OAA′, OBB′, OCC′ be *any* three chords, the centre of the conic for which the triangles ABC, A′B′C′ are reciprocal is always at the *same* point. ... (4.41)

*The necessary and sufficient condition that Γ shall be a parabola is that A″B″C″ should be collinear.** ... (4.5)

The condition is necessary. For let Γ be a parabola. Then O′ is at infinity by (4.4). So Σ, Σ' are either both parabolas or both hyperbolas. If both be parabolas, then the polar axis which is their common chord must be parallel to their axes. But then the parabola Σ'' touches at O a line parallel to its axis. That is, Σ'' degenerates into the polar axis and another line parallel to it. The other line must contain A″, B″, C″.

If both Σ, Σ' be hyperbolas, then the polar axis which is their common chord must be parallel to an asymptote. But then the

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* The necessity of this condition is stated in Qn. 1651 (*a*) (Vide ***The Math. Student*** Vol. I, No. 2).

hyperbola Σ'' touches at O a line parallel to one of its asymptotes. That is, Σ'' degenerates into the polar axis and another line which, as before, should contain A'', B'', C''.

To prove that the condition is sufficient, we observe that since A'', B'', C'' are collinear, Σ'' must degenerate into the line $A''B''C''$ and the polar axis, by (3.4). If $A''B''C''$ and the polar axis are parallel then Σ, Σ' are both parabolas and OO' their common chord being parallel to their axes, O' is at infinity and Γ is a parabola. If $A''B''C''$ and the polar axis be not parallel, then Σ, Σ' are both hyperbolas and OO' their common chord *i.e*, the polar axis) being parallel to an asymptote, O' is again at infinity, and as before Γ is a parabola.

It may be noted that the result applies only to the case where O is at a finite distance.

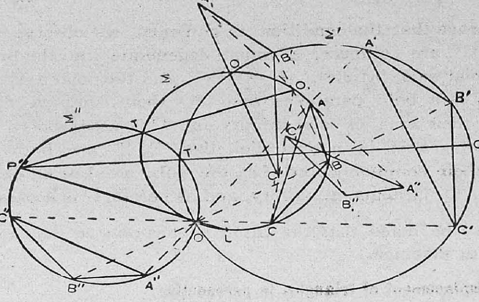
5. Displacement of triangles in perspective.

If the triangle $A'B'C'$ be given a translatory motion represented by OP where P is any point on Σ'', its perspectivity with the triangle ABC is maintained ; the new centre of perspective O_1 is also on Σ ; the triangle corresponding to the triangle $A''B''C''$ and the s. s. p. conics associated with the three triangles are all entirely unaltered in shape and size, being in fact the original ones merely displaced. ... (5.1)

To prove this, let Σ, Σ'' ; Σ', Σ'' respectively cut again in T, T'. (Refer to Fig. 1 next page).

Then, taking *any* point P on Σ'', let PT, PT' cut Σ, Σ' respectively in O_1, O_1'. Then from (2.1), PA'', PB'', PC'' are respectively parallel to O_1A, O_1B, O_1C, as well as to $O_1'A'$, $O_1'B'$, O_1C'. So if parallels to OP through A', B', C' cut O_1A, O_1B, O_1C in A_1', B_1', C_1', then each of the pairs of triangles $A'AA'_1$, $OA''P$ is congruent, since $A'A = OA''$, etc. So $A'A'_1 = B'B' = C'C'_1 = OP$. Also the triangles $A'B'O'_1$, $A_1'B_1'O_1$ are congruent and similarly placed, since $A'B' = A_1'B_1'$, etc., and $O_1A'_1$ is parallel to $O_1'A$, etc. ; so $O'_1 O_1 = OP$. In fact the tetrad $O_1A_1'B_1'C_1'$ may be considered as got by displacing the tetrad $O_1'A'B'C'$ by the vector $O_1'O_1$ or OP. Also since $A_1'A = PA''$, etc., from the pairs of congruent triangles $A_1'AA'$, $PA''O$, etc., if A_1'', B_1'', C_1'' be taken on O_1A, O_1B, O_1C so that $A_1'A = O_1A''$, etc., then the tetrad $O_1A_1''\,B_1''C_1''$ is got by displacing

the tetrad $PA''B''C''$ along the vector PO_1. So, if Σ' and Σ'' be respectively displaced along the vectors OP and PO_1, they should



[The figure is drawn for the case when Σ and Σ' are circles but the reasoning is quite general]

FIG. 1.

circumscribe $O_1A_1'B_1'C_1'$ and $O_1A_1''B_1''C_1''$ respectively. Since Σ is *s. s. p.* to Σ', Σ'' in their new positions also, it follows, by (3.3), that these three are the *s. s p.* conics in the new position. Hence the result.

The triangle ABC and Σ are fixed; the triangle $A'B'C'$ and Σ' receive a translation along OP; and the triangle $A''B''C''$ and Σ'' are displaced by PO_1. In fact the *s. s. p.* conics are simply (rigidly) *carried along* by the respective triangles Also, if PO cut Σ again in L,

The centre of Γ in the new position is L. ... (5.11)

For, by (4.4) and (3.4), the centre required is the point where the polar axis (in the new position) cuts Σ in addition to O_1. But, by (3.4), this polar axis is the tangent at O_1 to Σ'' (in the new position). Also the point in the old position of Σ'' displaced to O_1 is P. Hence the required centre is the other end of the chord (of Σ) through O_1 parallel to the tangent at P to Σ''. Since, by (2.1), PO contains this point, the result follows.

We can also show that :—

The envelope of the polar axis is a fixed conic s. s. p. to and concentric with Σ, ... (5.12)

Project Σ, Σ', Σ'' into circles. Let Fig. 1 be taken as that for the projections. Let H, H$'$, H$''$ be the centres of Σ, Σ', Σ''.

Since OO$'$ touches Σ'' at O, O_1O' and PO or OL are parallel by (2.1). So $O_1L =$ OO$'$. Hence the polar axis i nthe new position (*i.e.*, O_1L) touches the circle with centre H and radius the distance of H from OO$'$. The diameter of this circle is accordingly equal to the chord intercepted by Σ on H$''$O. Hence the result. Since HH$'$ bisects OO$'$ and is parallel to H$''$O, it also follows that the new conic is got by reducing each semi-diameter of Σ in the ratio of the chord intercepted by Σ on the diameter of Σ'' through O, the diameter, of Σ on the line of centres of Σ, Σ'.

Since the envelope reduces to a point when the chord reduces to a point, and since when H$''$O touches Σ, HO touches Σ'', it follows that :—

The polar axes are concurrent if the diameter of Σ (*or* Σ'') *through a common point touches* Σ'' (*or* Σ). ... (5.13)

If OABC, O$'$A$'$B$'$C$'$ be each concyclic, the centre of Γ *in each displaced position is at the same distance from the corresponding centre of perspective.* ... (5.14)

It also follows that if in one position of the triangle A$'$B$'$C$'$, Γ is a parabola, then, in *any* other position also Γ is *still* a parabola with a *parallel* axis provided the centre of perspective in each case is not at infinity. (5.2).

For, from (4.5), A$''$B$''$C$''$ are collinear in the first position and from (5.1), they are still collinear in the second position. Consideration of Σ'' (which in this case breaks up into the line A$''$B$''$C$''$ and the polar axis) shows by (5.1) that the polar axis and the line A$''$B$''$C$''$ maintain their directions. But the polar axis determines the direction of the axis of Γ. Hence the result.

6. Discussion of Question 1648 of Mr. V. Ramaswamy Aiyar.

Question 1648. (*V. Ramaswamy Aiyar.*) :—Two triangles ABC, A$'$B$'$C$'$ are in perspective. If A$'$B$'$C$'$ be moved rigidly and without rotation, so as to be constantly in perspective with ABC, show that the locus of any point P of its plane (of its *body*, if you please) is, in general, a conic.

In what case or cases will the locus of P be a line, apart from the obvious case where the centre of perspective O is at infinity when P can move straight towards O ?

(5.1) indicates that any point P on Σ'' (the P in the question is taken at O) satisfies the conditions of the problem. It only remains to show that *any* point P satisfying the condition lies on Σ''. Now since OP and OA$''$ are respectively parallel and equal to $A'A'_1$ and $A'A$ it follows that AA'_1 is parallel to $A''P$ and likewise BB'_1 to $B''P$ and CC'_1 to $C''P$. Thus the parallels to PA'', PB'', PC'' through A, B, C respectively are concurrent. [Refer to Fig. 1.]

Take the conic S'' through O, A'', B'', C'', P ; and through O, A, B draw S *s.s.p.* to S''. Draw the chord AO_2 of S parallel to PA''. Then, by (2.1), O_2B is parallel to PB''. So O_2 must be the point of concurrence of the parallels through A, B, C. Since the join of O_2 to the intersection of S and OC'' must be parallel to PC'' or O_2C, it follows that S passes through C also. Hence by (3.2) $S \equiv \Sigma$, $S'' \equiv \Sigma''$. Hence follows the first part.

We may add that the locus of P is the conic *circumscribing $A''B''C''$ and touching the polar axis.* By (24) its semi-diameter through O is equal to the distance between the centres of Σ, Σ'. ... (6.1)

The form of Σ, Σ' determines the form of Σ''. For example :—

If OABC, $OA'B'C'$ be each a cyclic quadrilateral, the locus of P is a circle. (6.11)

If only a single pair of corresponding sides, say AB, $A'B'$ be parallel, Σ'' breaks up into $A''B''$ and OC. (6.12)

If A'', B'', C'' be collinear, Σ'' breaks up into the line $A''B''C''$ and the polar axis. (6.13)

If O is the ortho-centre of the triangle ABC (or $A'B'C'$), Σ'' is a rectangular hyperbola breaking up into a pair of perpendicular straight lines if in addition A'', B'', C'' are collinear. (6.14)

(6.12), 6.13) provide the solution for the second part. In each case the locus of P is a *pair* of lines. Since $A_1'A$, $B_1'B$, $C_1'C$ are respectively parallel to PA'', PB'', PC'', it follows that :—

When A'', B'', C'' are collinear, there are two types of displacements. For *any* displacement OP where P is on the line $A''B''C''$, the new centre of perspective is at infinity, A_1', B_1', C_1' respectively lying on the parallels to $A''B''C''$ through ABC ; for all *other* dis-

placements, *i.e.*, those parallel to the polar axis, the new centre is *not* at infinity. ... (6.15)

This result may be useful to explain the positions in (5.2)

The case when O is at infinity.

We shall prove that in this case displacements *other* than those parallel to AA′ are possible.

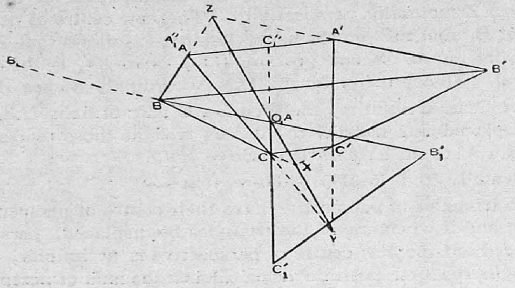


FIG. 2.

Draw BO_1, CO_1 parallel to B′A′, C′A′ respectively (Fig. 2.)* Displace the triangle A′B′C′ by $A'O_1$. Then A_1' is at O_1, and B_1', C_1' are respectively on BO_1 and CO_1. Hence O_1 is the new centre of perspective.

It is easy to see that A_1'' is at **A** and that O_1B, O_1C cut AA′ in B_1'', C_1''. The centre of perspective being at a vertex, the family (F_1') should be taken as the one circumscribing the triangle $A'_1B'_1C'_1$ and *touching* A'_1A. The locus in this new position is accordingly the line $A''_1B''_1C''_1$ and the polar axis. But since every possible position of A′B′C′ is a possible position of $A'_1B'_1C'_1$ and conversely, it follows that the locus in the original position is also the same for displacements to be given to the point A′. We can show that

The polar axis in the second position is the axis of perspective in the first. (6.21)

For O_1 is the centre of Γ (in the first position). Because the triangle O_1BC and its reciprocal triangle are in perspective, and O_1B and the polar of C (*i.e.*, A′B′) are parallel and O_1C and the polar of B (*i.e.*, A′C′) are parallel, it follows that the polar of O_1 is parallel to

* The triangle BO_1C is fixed for all displacements of the triangle A′B′C′ parallel to AA′. In the figure the triangle A′B′C′ is drawn for the *particular* position for which the axis of perspective *contains* O_1.

BC. So $A'O_1$ is a diameter. Since XYZ itself is also a diameter, O_1 is the centre. Again, denoting by $\Omega_{A'B'}$, $\Omega_{A'C'}$ the points at infinity on A′B′ and A′C′, the triangles $BZ\Omega_{A'B'}$ and $C\Omega_{A'C'}C'$ are reciprocal for Γ (the line $C\Omega_{A'C'}$ is only the diameter CO_1). Hence BC and the parallels through Z and C′ to A′C′ and A′B′ respectively, are concurrent, say at C_1'. Since $C'C_1' = A'Z$, the displacement of the triangle A′B′C′ by A′Z maintains perspectivity. The new centre of perspective is the point B, and this not being at infinity, it follows that Z lies on the polar line in the second position (*i.e.*, where O_1 is the centre of perspective). Hence the result (6.21). Accordingly we see that :—

When O is at infinity, the locus is a pair of lines, *i.e.*, A′ can receive displacements parallel to AA′ as well as those represented by vectors from A′ to the axis of perspective XYZ. (6.22)

Incidentally from (6.21), it follows that :—

If two triangles in perspective have their centre of perspective at a finite point and if when one of the triangles be displaced, perspectivity is maintained but the new centre of perspective is at infinity, then the polar axis in the first position is parallel to the axis of perspective in the second position. (6.23)

Geometric determination of the possible displacement of the triangle A′B′C′ in any given direction.

When Σ'' is a circle or a pair of lines, the construction is obvious. The following covers the general case.

Let O′A′, O′A cut Σ, Σ' respectively in P, P′. Let OP cut Σ' in P′. Then, by (2.1), PB, PC are respectively parallel to A′B′, A′C′ and (P′) B′, (P′) C′ are respectively parallel to AB, AC. Hence P, (P′) can be constructed.* Also (P′) P′ is parallel to AP and hence P′ and therefore P″ where P′P = OP″ can be got. Now O, A″, B″, C″, P″ being five points on Σ'', the point X″ on it lying on any specified line OX can be got from the equality of the cross-ratios, O (X″ A″ B″ C″) and P″ (X″ A″ B″ C″) by the usual methods. OX″ is the required displacement.

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* This method of constructing P, Q, R and (P′), (Q′), (R′) defines Σ Σ' irrespective of the position of O. Also the locus, say of A′ (for displacements) is Σ'' displaced by OA′. So even when O is at infinity we get Σ, Σ' and the locus of A′ should be *s. s. p.* to them. But since the ends of the vectors from A′ equal to A′A, B′B, C′C all lie on A′A, the locus of A′ should degenerate into a *pair* of lines parallel to the asymptotes of Σ, Σ' confirming the preceding discussions of the case when O is at infinity.

THE ARCHIMEDIAN SOLIDS.

By P. K. Kashikar,

Wilson College, Bombay.

1. An Archimedian solid is a convex polyhedron, whose faces are regular polygons, not all similar, and whose solid angles are all congruent.

The present paper contains, what is believed to be a new proof of the theorem, that there cannot be more than fifteen different types of these solids and also a simple method for the construction of card-board models.

As the figure is convex, the plane-angle sum at a vertex is less than 360°; and hence

(1) The polygons at a vertex cannot belong to more than three different species, for the least angle-sum for four species is $60° + 90° + 108° + 120° = 378°$, and

(2) the number of polygons at a vertex must not exceed five, for if it be six, the least value of the angle-sum would be $5 \times 60° + 90° = 390°$.

2. Theorem A. If there are three polygons at a corner and the number of sides in one of these is odd, then the two remaining polygons must be similar and must have an even number of sides.

Let P_α, P_β and P_γ denote regular polygons of angles, α, β and γ. Let A_1A_2 and A_2A_3 be two consecutive edges of P_α and let the other polygon standing on A_1A_2 be P_γ. Then P_β must be the polygon on A_2A_3, P_γ the polygon on A_3A_4 and so on. But this is possible only if P_α has an even number of sides. If this number be odd, then we must have $\beta = \gamma$.

Theorem B. If there are three polygons at a corner and two of these are similar, then each of these two polygons must have an even number of sides, (for otherwise there will be a

contradiction to Theorem A′, and if all the polygons are dissimilar, then each of them must have an even number of sides.

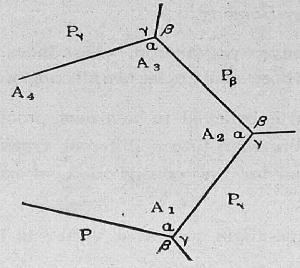

FIG. 1.

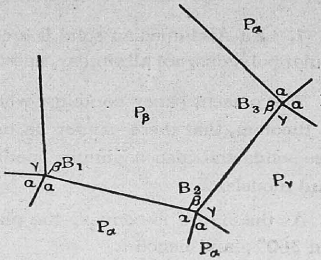

FIG. 2.

THEOREM C. If there are four polygons at a corner, then two at least must be similar (otherwise the angle-sum will exceed 360°) and these cannot be adjacent if one of the remaining polygons has an odd number of sides.

Let the polygons be P_α, P_α, P_β, and P_γ and let P_β be the polygon with an odd number of sides. If possible let the order of the angles at B_1 be $\beta\gamma\alpha\alpha$; then the order at B_2 must be $\beta\alpha\alpha\gamma$. Thus the polygon on the other side of B_1B_2 is P_α; the polygon on the other side of B_2B_3 is P_γ, and so on. But this is impossible when P_β has an odd number of sides.

THEOREM D. If there are four polygons at a vertex and only two of these are triangles, then the two remaining polygons must be similar and non-adjacent.

For if P_α is a triangle, the two equal angles α (= 60°) may be (1) adjacent or (2) opposite. But going round the corners in the order $A_1A_2A_3$ it will be found that in either case, the polygon on the other side of A_3A_1 will not be regular unless two of the angles α, β, γ are

equal. But if there are only two triangles, α cannot be equal to β or γ. Therefore $\beta = \gamma$, *i.e.*, $P_\beta \equiv P_\gamma$.

Theorems A, B, C, D together with Euclid's theorem that the sum of the plane angles at a vertex is less than 360° (which will be referred to as Theorem E) enable us to find out the possible types of the Archimedian solids. We consider the polygons at a corner and classify the figures as follows :

I. *Solids with four triangles at a vertex.*

There can be only one more polygon which may be a square or a pentagon (E). The two corresponding figures may be conveniently denoted by the symbols $3_4 4_1$ (the Snub Cube) and $3_4 5_1$ (the Snub Dodecahedron) there being four triangles and one square at a vertex of the first figure and four triangles and one pentagon at a vertex of the second.

II. *Solids with three triangles at a vertex.*

There can be only one more polygon and it may have any number of sides (E). This figure is called the Semiregular Prismoid, its symbol being $3_3 n_1$.

III. *Solids with two triangles at a vertex.*

The number of remaining polygons cannot be one (B). Therefore there must be two more polygons and these must be both squares or both pentagons (B). The two corresponding figures are the Cuboctahedron $3_2 4_2$ and the Icosidodecahedron $3_2 5_2$.

IV. *Solids with one triangle at a vertex.*

(*a*) There may be two more polygons or (*b*) there may be three.

(*a*) If there are only two, they must be similar polygons with an even number of sides (A). Therefore they must be both squares, both hexagons, both octagons or both decagons (E). Thus we get the solutions $3_1 4_2$. (The triangular prism included in the general class of prisms V); $3_1 6_2$ (the truncated Tetrahedron); $3_1 8_2$ (the Truncated Cube) and $3_1 10_2$ (the Truncated Dodecahedron).

(*b*) If there are three more polygons, they cannot be all dissimilar (E). Therefore they may be (1) all similar, in which case each must be a square (E) or (2) only two of them may be similar (C) in which case we must have two squares and one pentagon (E). The two corresponding figures are $3_1 4_3$ (the Small Rhombicuboctahedron) and $3_1 4_2 5_1$ (the Small Rhombicosidodecahedron).

V. *When there are no triangles.*

The greatest number of squares at a vertex is two; and in this case there will be only one more polygon (E) which may have any number of sides. This figure is the Semiregular Prism $4_2 n_1$.

If there is only one square (but no triangle), there can be only two more polygons (E). If these are similar, they must be both hexagons (B, E) and we get the figure called the truncated Octahedron $4_1 6_2$. If the two polygons are dissimilar, they must have an even number of sides (B); and there may be one hexagon and one octagon, or one hexagon and one decagon, other combinations being impossible (E). The two corresponding figures are the Great Rhombicuboctahedron $4_1 6_1 8_1$ and the Great Rhombicosidodecahedron $4_1 6_1 10_1$.

VI. *When there are no triangles and no squares.*

The greatest number of pentagons at a corner is two (E), but then there will be only one polygon more, and this combination is impossible (B). Hence we must have only one pentagon and two hexagons (B, E). This is the Truncated Icosahedron $5_1 6_2$.

VII. *When there are no triangles, squares or pentagons.*

The least value of the angle-sum will be greater than $360°$ and no figure is possible in this case.

This proves that there are only fifteen possible types of Archimedian solids,* *viz.*, (1) $3_4 4_1$, (2) $3_4 5_1$, (3) $3_3 n_1$, (4) $3_2 4_2$, (5) $3_2 5_2$, (6) $3_1 6_2$, (7) $3_1 8_2$, (8) $3_1 10_2$, (9) $3_1 4_3$, (10) $3_1 4_2 5_1$, (11) $4_2 n_1$, (12) $4_1 6_2$, (13) $4_1 6_1 8_1$, (14) $4_1 6_1 10_1$, (15) $5_1 6_2$.

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* It may be shown that if F, V, E be the numbers of faces, vertices, and edges of a polyhedron at each vertex of which meet a polygons with α sides, b polygons with β sides and c polygons with γ sides, then

$$V = 2/(1 + \Sigma\, a/\alpha - \tfrac{1}{2}\Sigma\, a)\,;\; E = \tfrac{1}{2} V(a+b+c)\,;\; F = 2+E-V = 2+\tfrac{1}{2} V(a+b+c-2)$$

A table of values is given in the article on Geometrical Solids, ***Encyclopaedia Britannica***, (Fourteenth Edition).

Each of these figures has got a circumscribed sphere and the radius of this sphere can be calculated after determining the dihedral angles by the methods of spherical trigonometry.

It is easy to construct card-board models of these figures by the method described below. The card-board must be flexible and not very thick. The ordinary brown card-board is not suitable, but the paper of visiting cards may be used.

Let P be the polygon which occurs the least number of times (p say) in the whole figure. The arrangement of the other dissimilar polygons surrounding P may be found by considering the form of the solid angles at the corners of P. On the surface of the solid, P and the surrounding polygons will form a cap-like figure. Let this be developed into a plane figure, and drawn on a piece of card-board. Join the other extremities of each pair of edges which meet at a corner of P (but do not belong to P) and cut the paper along the boundary of the figure so formed. It will be found that there are certain triangular areas between two neighbouring polygons, which do not belong to the solid; but these are not to be cut off. Now, if the central polygon P has n sides, the surrounding polygons may be divided into n groups, one group belonging to each side of P.

These groups are to be separated from one another (but not from the central polygon P) by cutting the paper along the proper edges, and creases are to be formed by folding the paper along all the lines in the figure. Then the triangular part outside each group, is to be pasted over the adjacent polygon of the next group (on the concave side of the figure). When this is done for all the groups, a cap will be formed, and p such caps fitted together and pasted one over the other, will form the required polyhedron. In pasting one part of the figure over another, it is necessary to fix the two parts in the proper position until the paste is dry; and for this purpose clips will have to be used. Again, before folding the paper along any line, it should be ruled along that line with the sharp point of the divider.

By way of illustration, let the required polyhedron be $3_1 4_2 5_1$ (the Small Rhombicosidodecahedron).

Fig. 3 shows the plane development of a cap. The superfluous triangular areas are AP_1T_8, BQ_1P_8, etc. The paper is to be cut

along the lines AP_1, BQ_1, CR_1, DS_1 and ET_1 and to be folded along all the other lines in the figure. The triangle AP_1T_3 is then to be pasted over the square AP_1P_2B; the triangle BQ_1P_3 over the square BQ_1Q_2C and so on. Twelve such caps can be fitted together to form the required polyhedron.

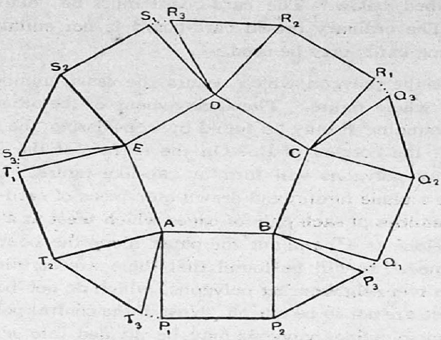

FIG. 3.

This method is applicable to any convex polyhedron, but can be conveniently modified when some of the surrounding polygons are very large and also when the number of vertices is smaller than the number of faces. In this latter case, we form pyramidal caps having the shape of the solid angle of the figure, and these caps are fitted together. With some modifications the method can also be used to construct card-board models of the concave polyhedra obtained by stellating the convex regular solids.

Other materials which may be used to construct models are celluloid wires, sticks, strings, clay, wood, etc. But celluloid is costly, wire and strings do not give coloured models, and it is difficult to work with wood or clay.

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26. 'Life and cosmos are periodic continued fractions, and hence both are irrational.'

'*Numerology*,' by E. T. BELL.
(Per Mr. A. A. KRISHNASWAMI AYYANGAR).

AN EXTENSION OF HEILBRONN'S CLASS-NUMBER THEOREM.

BY S CHOWLA.

Let $h(d)$ denote the number of primitive classes of binary quadratic forms of negative discriminant d. Heilbronn has recently shown that

THEOREM A.

$$h(d) \to \infty \quad as \; -d \to \infty.$$

This result was conjectured by Gauss.

I have succeeded in proving the following extension (also conjectured by Gauss) of Theorem A.

THEOREM B. *If* t *denotes the number of different prime factors contained in* d, *then*

$$\frac{h(d)}{2^t} \to \infty, \quad as \; -d \to \infty.$$

We use the method used by Heilbronn to prove Theorem A. The important new weapon is the following theorem of Pepin:

THEOREM. *Let* d *be quadratfrei and*

$$a/(x^2 + d), \qquad (a, 2d) = 1.$$

Then

$$a^{2p} = x^2 + dy^2 \qquad (y \neq 0$$

where p *is the number of classes in the principal genus of reduced primitive non-equivalent binary quadratic forms of negative determinant* —d.

This theorem is quoted by Dickson, *History of the theory of numbers*, Volume 3, page 58. Details of the proof of Theorem B will appear elsewhere.

[Received July 18, 1934.]

NOTES AND DISCUSSIONS.

The Editor welcomes for publication under this heading, brief discussions of interesting problems, critical comments, and suggestions likely to be helpful in the class-room.

On the Radius of a circle and the condition of orthogonality of two given circles.

In this note I obtain expressions for the radius of a circle whose equation is given in areal co-ordinates, from the theorem that the numerical value of the power of the centre with respect to the circle is equal to the square of the radius.

Let $$\phi(x, y, z) \equiv \Sigma\, ux^2 + 2\,\Sigma\, fyz = 0,$$

be the equation of a circle in areal co-ordinates, so that

$$\frac{v+w-2f}{a^2} = \frac{w+u-2g}{b^2} = \frac{u+v-2h}{c^2} = k \quad \text{(say)}\ ;\ \ldots \quad (1)$$

then, the power of the point $P(x_1, y_1, z_1)$ with respect to the circle is

$$\phi(x_1, y_1, z_1)\,/\,k \qquad \ldots \quad (2)$$

If the point $P(x_1, y_1, z_1)$ is the centre, then

$$ux_1 + hy_1 + gz_1 - \lambda = 0; \quad hx_1 + vy_1 + fz_1 - \lambda = 0;$$
$$gx_1 + fy_1 + wz_1 - \lambda = 0; \quad x_1 + y_1 + z_1 - \lambda = 0$$

Hence $$\begin{vmatrix} u & h & g & \lambda \\ h & v & f & \lambda \\ g & f & w & \lambda \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0, \text{ i.e., } \begin{vmatrix} u & h & g & 1 \\ h & v & f & 1 \\ g & f & w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} \lambda + \Delta = 0 \ \ldots \quad (3)$$

where Δ is the discriminant of ϕ.

The determinant in the last equation equals

$$\frac{1}{4}\begin{vmatrix} 2u & u+v-kc^2 & w+u-kb^2 & 1 \\ u+v-kc^2 & 2v & v+w-ka^2 & 1 \\ w+u-kb^2 & v+w-ka^2 & 2w & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = -4k^2 S^2$$

where S is the area of the triangle of reference. Therefore $\lambda = \Delta/4S^2k^2$.

Also $$\phi\ (x_1, y\ , z_1) = \tfrac{1}{2}\ \Sigma\ x_1 \frac{\partial \phi}{\partial x_1} = \lambda.$$

Hence, if ρ be the radius of the circle,

$$-\rho^2 = \frac{\lambda}{k} = \frac{\Delta}{4S^2k^3} \quad \ldots \quad (3)$$

On substituting for Δ and using (1), this result may also be written in the form

$$\rho^2 = \frac{\Sigma\, u^2a^2 - 2\ \Sigma\ bc\cos A\ vw - 2\ abc\, k\ \Sigma\ ua\cos A + k^2a^2b^2c^2}{16S^2k^2} \quad \ldots \quad (4)$$

Condition of orthogonality of two circles.

Let $\phi_r\ (x, y, z) \equiv \Sigma\ u_r x^2 + 2\ \Sigma f_r\ yz \equiv 0 \qquad (r = 1, 2)$

be two circles, so that

$$\frac{v_r + w_r + 2f_r}{a^2} = \frac{w_r + u_r - 2g_r}{b^2} = \frac{u_r + v_r - 2h_r}{c^2} = k_r.$$

Suppose (x_r, y_r, z_r) is the centre and ρ_r is the radius of the circle ϕ_r; then

$$-\rho_1{}^2 = \frac{\phi_1\ (x_1, y_1, z_1)}{k_1}; \qquad d^2 - \rho_2{}^2 = \frac{\phi_2\ (x_1, y_1, z_1)}{k_2} \quad \ldots \quad (5)$$

where d is the distance between the centres of the circles. If the circles cut orthognally

$$d^2 = \rho_1{}^2 + \rho_2{}^2 \quad \ldots \quad (6)$$

so that by (5), the centre $(x_1\ y_1\ z_1)$ of ϕ_1 lies on the radical circle

$$\phi_1/k_1 + \phi_2/k_2 = 0.$$

Similarly (x_2, y_2, z_2) lies on this circle. Consequently if the circles ϕ_1 and ϕ_2 cut orthogonally, the radical circle of the given circles passes through the centres of both the circles.* Since it also passes through the common points of the circles, the line joining the centres of ϕ_1 and

* Prof. Rao's note in the *Bulletin of the Calcutta Mathematical Society*, Vol. XXI, 1929, pp. 195-6.

ϕ_2 is a diameter of the radical circle. Thus if ρ_3 be the radius of the radical circle, the condition of orthogonality of the circles ϕ_1 and ϕ_2 can be put in the form $\rho_1^2 + \rho_2^2 = 4\rho_3^2$. Hence from (4)

$$\frac{1}{16S^2k_1^2}\left[\Sigma u_1^2a^2 - 2\,\Sigma\, bc\cos A\, v_1w_1 - 2\,abc\,k_1\,\Sigma\, u_1a\cos A + k_1^2a^2b^2c^2\right]$$

$$+\frac{1}{16S^2k_2^2}\left[\Sigma u_2^2a^2 - 2\,\Sigma\, bc\cos A\, v_2w_2 - 2\,abc\,k_2\,\Sigma\, u_2a\cos A + k_2^2a^2b^2c^2\right]$$

$$=\frac{1}{16S^2}\left[\Sigma\left(\frac{u_1}{k_1}+\frac{u_2}{k_2}\right)^2a^2 - 2\,\Sigma\, bc\cos A\left(\frac{v_1}{k_1}+\frac{v_2}{k_2}\right)\left(\frac{w_1}{k_1}+\frac{w_2}{k_2}\right) - 4abc\,\Sigma\left(\frac{u_1}{k_1}+\frac{u_2}{k_2}\right)a\cos A + 4a^2b^2c^2\right].$$

This reduces to

$$\Sigma\, a^2u_1u_2 + a^2b^2c^2\,k_1k_2 = \Sigma\, bc\cos A\,(v_1w_2 + v_2w_1) + abc\,\Sigma\,(u_1k_2 + u_2k_1)\,a\cos A.$$

The advantage of the method is that while using the usual condition of orthogonality (6), we have avoided the calculation of d and used instead the fact that d is the diameter of the radical circle.

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Recurrence Formulae for $J_n(x)$.

In this note I suggest proofs of the well-known relations (i) to (iv) connecting Bersel functions, which are simpler than those usually given in usual text-books.

(i) $2n\,J_n = x\,(J_{n-1} + J_{n+1})$

(ii) $2\,\frac{d}{dx}J_n = J_{n-1} - J_{n+1}$

(iii) $x\,\frac{d}{dx}J_n = n\,J_n - x\,J_{n+1}$

(iv) $x\,\frac{d}{dx}J_n = -\,n\,J_n + x\,J_{n-1}$

To prove (i) we start with the definition of $J_n(x)$, namely,

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\phi - x\sin\phi)\,d\phi \quad \ldots \quad (1)$$

[See Ganesh Prasad's *Spherical Harmonics*, Article 64.]

$$\therefore \quad J_{n+1} = \frac{1}{\pi}\int_0^\pi \cos(\overline{n+1}\,\phi - x\sin\phi)\,d\phi$$

$$= \frac{1}{\pi}\int_0^\pi \cos(\psi + \phi)\,d\phi \quad \ldots \quad (2)$$

$$J_{n-1} = \frac{1}{\pi}\int_0^\pi \cos(\overline{n-1}\,\phi - x\sin\phi)\,d\phi$$

$$= \frac{1}{\pi}\int_0^\pi \cos(\psi - \phi)\,d\phi \quad \ldots \quad (3)$$

$$J_n = \frac{1}{\pi}\int_0^\pi \cos\psi\,d\phi \quad \ldots \quad (4)$$

where $\psi = n\phi - x\sin\phi$ ... (5)

From (2) and (3) we get by addition

$$J_{n+1} + J_{n-1} = \frac{2}{\pi}\int_0^\pi \cos\psi\cos\phi\,d\phi \quad \ldots \quad (6)$$

Again from (5), $d\psi = (n - x\cos\phi)\,d\phi$, so that we get

$$\frac{1}{\pi}\int_0^\pi \cos\psi\,d\psi = \frac{n}{\pi}\int_0^\pi \cos\psi\,d\phi - \frac{x}{\pi}\int_0^\pi \cos\psi\cos\phi\,d\phi.$$

But as $\int_0^\pi \cos\psi\,d\psi = 0$ and $\frac{1}{\pi}\int_0^\pi \cos\psi\,d\phi = J_n$ by (4),

it follows that

$$n\,J_n = \frac{x}{\pi}\int_0^\pi \cos\psi\cos\phi\,d\phi. \quad \ldots \quad (7)$$

The result (i) follows from (6) and (7).

Again, on differentiating (1) with respect to x we get

$$\frac{d}{dx} J_n = \frac{1}{\pi} \int_0^{\pi} \sin (n\phi - x \sin \phi) . \sin \phi \, d\phi = \frac{1}{\pi} \int_0^{\pi} \sin \psi \sin \phi \, d \qquad \dots \quad (8)$$

But, from (3) and (2) by subtraction we get

$$J_{n-1} - J_{n+1} = \frac{2}{\pi} \int_0^{\pi} \sin \psi \sin \phi \, d\phi$$

whence, using (8), we get result (ii).

To prove the relations (iii) and (iv) we start with (ii) which we have just now proved.

$$2x \frac{d}{dx} J_n = x J_{n-1} - x J_{n+1} \qquad \dots \quad (9)$$

Again

$$2n J_n = x J_{n-1} + x J_{n+1} \qquad \dots \quad (10)$$

Eliminating J_{n-1} between (9) and (10) we get the result No. (iii), while eliminating J_{n+1} between (9) and (10) we get the result No. (iv).

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RATAN LAL GUPTA, M.Sc.

Lecture Notes.

On the identity $(\sum_1^N n)^2 = \sum_1^N n^3$.

The usual text-book proofs of this result are by the method of induction. The result may, however, be directly verified as follows :—

LEMMA 1.
$$(a + b + c + \dots l + k)^2 = a^2 + b^2 + c^2 + \dots k^2 + 2a(b + c + \dots k) + 2b(c + d + \dots k) + \dots \dots \dots \dots + 2lk.$$

LEMMA 2. $m^2 + 2m(\overline{m-1} + \overline{m-2} + \dots 2 + 1) = m^3$.

Now

$$(1 + 2 + 3 \dots n)^2 = (n + \overline{n-1} + \overline{n-2} + \dots 2 + 1)^2$$
$$= n^2 + 2n\,(\overline{n-1} + \overline{n-2} + \dots 2 + 1)$$
$$+ (n-1)^2 + 2\,(n-1)\,(\overline{n-2} + \overline{n-3} + \dots 2+1)$$
$$+ (n-2)^2 + 2\,(n-2)\,(\overline{n-3} + \overline{n-4} + \dots 2+1)$$
$$+ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$
$$+ 2^2 + 2.2.(1)$$
$$+ 1^2 \qquad \text{by Lemma 1}$$
$$= n^3 + (n-1)^3 + (n-2)^3 + \dots 2^3 + 1 \text{ by Lemma 2.}$$

Trichy. S. SURYANARAYANA.

Multiplication of numbers ending in 5.

It is useful to notice that the product of two numbers ending in 5, say $a5$ and $b5$ (where a and b are any rows of figures) is given by

$$a5 \times b5 = \left\{ a \times b + \left[\frac{a+b}{2}\right] \right\} c$$

where c stands for the figures 25 or 75 according as $a \sim b$ is even or odd, and $[x]$ denotes the greatest integer in x.

Thus $$1175 \times 1325 = \left[117 \times 132 + \left[\frac{117+132}{2}\right] \right\} 25$$
$$= 1556875.$$

Vizagapatam. K. SUBBA RAO.

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27. 'A problem may be solved by merely human intelligence, but a theorem (derived from '*Theos*' God and '*res*' a thing) requires divine assistance.'

Quoted by DEAN INGE in '*Lay Thoughts of a Dean*.'

(Per Mr. A. A. KRISHNASWAMI AYYANGAR).

28. 'Our account with our environment may be represented by a vulgar fraction, the numerator being what we have and the denominator what we want. We may improve this either by increasing our numerator, which is the wisdom of the West, or by diminishing our denominator, which is the wisdom of the East. Which is the right method of solving the problem?' DEAN INGE.

(Per Mr. A. A. KRISHNASWAMI AYYANGAR).

Sundaram's Sieve for Prime Numbers.

Mr. S. P. Sundaram, of Satyamangalam, is a young mathematical student (a relation of mine) who, it appears, has not been able to get through his "Intermediate." He wrote to me recently giving me an interesting theorem of his, which may be stated as follows :—

Let us form a double series whose corner term is 4. The first row is an A. P. with common difference 3. The first column is the same as the first row. The second row is an A. P. with common difference 5; the third, with common difference 7; the fourth, with 9; and so on. The double series will thus be

4	7	10	13	16	19	...
7	12	17	22	27	32	...
10	17	24	31	38	45	...
13	22	31	40	49	58	...
16	27	38	49	60	71	...
19	32	45	58	71	84	...
...	...	...	...	...	...	...

Then it will be seen that the kth column series is the same as the kth row series, for all k.

Mr. Sundaram's Theorem is as follows :—*If* N *be any number of the natural series 1, 2, 3, 4, 5, ..., which occurs in the double series, then* $2N + 1$ *is a composite number. But if* N *does not occur in the double series, then* $2N + 1$ *is a prime number; and all the primes except 2 can be got in this manner.*

The Theorem strikes me as both interesting and beautiful; but, most of our readers, I think, will be able to prove it for themselves, if they try.

V. RAMASWAMI AIYAR.

Question 1571.

(V. RAMASWAMI AIYAR, M.A.) :—Let M be the foot of the perpendicular drawn from a focus S on the tangent at a fixed point P of a conic. If PP′, SS′ be perpendiculars drawn on a variable tangent of the conic, show that PP′.SS′ varies as MS'^2.

Solution by A. Ranganatha Rao.

Draw PQ perpendicular to SS′. Then the fixed circle Σ described on PS as diameter passes through M and Q. From a known property of conics, CM is parallel to HP, C being the centre and H the other focus of the conic. Therefore, CM bisects PS. Thus M, a point common to the circle Σ and the auxiliary circle of the conic, lies on their line of centres. Hence the two circles touch at M. If S′M cut the circle Σ again in R, then MR : MS′ = radius of the circle Σ : radius of the auxiliary circle = constant. Therefore RS : MS′ is also constant. Now,

$$\frac{PP'.SS'}{MS'^2} = \frac{QS'.SS'}{MS'^2} = \frac{RS'.MS'}{MS'^2} = \frac{RS'}{MS'} = \text{constant.}$$

Hence, PP′.SS′ varies as MS'^2.

Question 1574

(V. RAMASWAMI AIYAR, M.A.) :—In a given triangle ABC, AX is drawn perpendicular to BC and two points P, Q are taken thereon such that P, Q, B, C are mutually ortho-centric. Show that the pedal circles of P, Q with respect to ABC touch one another at X. Also if T be the point in BC whose pedal circle with respect to ABC touches those of P and Q at X, then TP, TQ envelope a hyperbola of which XA, XA′ are asymptotes and AA′ is a tangent where AA′ is the diameter through A of the circum-circle of ABC.

Solution by A. Ranganatha Rao.

Let P′, Q′ be the isogonal conjugates of P, Q with respect to ABC. P′, Q′ obviously lie on AA′. Let PQ′, P′Q meet in T_0 and let Ω be the point at infinity on AX.

Since $XP.XQ = BX.XC =$ constant, P, Q form a pair belonging to an involution range on XA. Now, to every point P on AX corresponds uniquely a point Q, and *vice versa*; and to a point Q on AX corresponds uniquely a point Q′, on AA′, and *vice versa*. Hence the envelope of PQ′ (and obviously also of QP′) is a conic Σ touching AX and AA′. The point of contact of Σ with AX must correspond to the point A on AA′. Now, if Q′ is at A, Q is at X and P is at Ω which is, therefore, the point of contact required. Hence XA is an asymptote of Σ. Again, the point of contact of Σ with AA′ must correspond to the point A on AX. Now, if P is at A, Q is at the ortho-centre of ABC, and Q′ is at the circum-centre O of ABC. Thus Σ touches AA′ as its mid. point. If P is at X, Q is at Ω and Q′ is at A′. Therefore XA′ is a tangent to Σ, the point of contact being Ω', say. Since Σ is an in-conic of the triangle AXA′ the points of contact with the sides being Ω, Ω', O, the lines $A\Omega'$, XO, $A'\Omega$ must be concurrent. Hence Ω' is the point at infinity on XA′, *i.e.*, XA′ is the other asymptote of Σ and therefore X is the centre of Σ.

Now, to solve the question, consider one position of P on AX. It is known that isogonal conjugates with respect to ABC are pairs of conjugate points with respect to the pencil of rectangular hyperbolas passing through the in- and ex-centres of ABC. Since PP′, QQ′ are isogonal conjugates, it follows by Hesse's theorem that A and T_0 are isogonal conjugates. Hence T_0 lies on BC. Again, consider the system of conics inscribed in the complete quadrilateral formed by the lines AX, AA′, PQ′, QP′. The centres of these conics lie on a line q, which passes through X, since Σ belongs to the system, and through the mid. points of PP′, QQ′, AT_0 since these are the point-pairs belonging to the system. The pedal circle of P with respect to ABC is the auxiliary circle of the in-conic of ABC, whose foci are P, P′. The centre of this pedal circle is, therefore, the mid. point of PP′. Thus the centres of the pedal circles of P, Q, T_0 lie on a line q passing through X which lies on the three pedal circles. Hence the three pedal circles touch at X and T_0 is no other than the point T. The envelope of TP, TQ is, as we have already seen, the conic Σ possessing the properties mentioned in the question.

Question 1593.

(B. B. BAGI) :—O is any point in the plane of a triangle ABC. BC, CA, AB, OA, OB, OC are of lengths a, b, c, a', b', c' and they make angles α, β, γ and α', β', γ' with a straight line respectively. Prove that there exists a triangle whose sides are proportional to aa', bb', cc' and whose exterior angles are λ, μ, ν where λ, μ and ν are given by

$$\left.\begin{aligned} \lambda &\equiv \gamma + \gamma' - \beta - \beta', \\ \mu &\equiv \alpha + \alpha' - \gamma - \gamma', \\ \nu &\equiv \beta + \beta' - \alpha - \alpha', \end{aligned}\right\} \text{mod. } (2\pi).$$

Give a geometrical construction for the triangle.

Solution by S. Ramaswami Iyer.

Construction. At A and B make angles BAE and ABE equal to $\angle$CAO and $\angle$ACO respectively to meet at E. Then the triangle EBO satisfies the condition of the question. For by construction, the triangles ABE, ACO are similar. Hence AB/AC = BE/CO, *i.e.*, BE = CC'/b.

Also AB/AE = AC/AO and $\angle$BAC = $\angle$EAO.

$\therefore$ the triangles A EO and ABC are similar. Hence AO/AC = EO/BC, *i.e.*, EO = aa'/b.

Hence BE : EO : BO = $cc'/b : aa'/b : b' = cc' : aa' : bb'$.

$$\begin{aligned} \angle EBO &= \angle EBA + \angle\ ABC - \angle OBC \\ &= \angle OCA + \angle ABC - \angle OBC \\ &= \beta - (\pi + \gamma') + \alpha - (\pi + \gamma) - (\overline{\alpha - \pi + \beta}) \\ &= (\beta + \beta') - (\gamma + \gamma') - \pi. \end{aligned}$$

Hence the exterior a ngle is

$$\pi - (\beta + \beta') + (\gamma + \nu') + \pi$$

$$= (\gamma + \gamma') - (\beta + \beta') \text{ mod, } 2\pi.$$

Similarly for the other exterior angles.

Question 1609.

(R. RAMANUJAN) :—Construct a triangle (geometrically) given the base, one of the base angles and the internal or external bisector of the vertical angle.

Solution by A. V. K. Krishna Menon.

With AB as base, describe the arc containing the angle equal to the vertical angle.

Let P be the middle point of the arc subtending the vertical angle.

Join PB and draw BQ at right angles to PB making BQ = half the given length of the bisector of the vertical angle.

With Q as centre and radius QB describe a circle. Let PQ produced cut this circle at R and PQ at S. With P as centre and radius PR describe a circle cutting the arc first drawn at C

Then ACB will be the required triangle.

Join PC cutting AB at T.

In the triangles BPT and PBC, $\angle P$ is common, and $P\hat{B}T = P\hat{C}A$

$= P\hat{C}B.$

$\therefore$ the triangels are similar and $\frac{CP}{BP} = \frac{BP}{PT}$.

$\therefore$ $BP^2 = CP.PT$. But $BP^2 = PS.PR$.

$\therefore$ $PC.PT = PS.PR$.

But $PC = PR$ [construction].

$\therefore$ $PS = PT$.

$\therefore$ $CT = SR$ = the given length of the bisector of the vertical angle.

When the external bisector is given, the same method applies. Only we have to take instead of P, its diametrically opposite point, and S instead of R.

Question 1610.

(S. MUKHOPADHYAYA) :—ABC is any triangle. BX and CX are, respectively, the internal (external) trisectors of the angles B and C nearest to BC; CY and AY are, respectively, the internal (external) trisectors of the angles C and A, nearest to CA; AZ and BZ are, respectively, the internal (external) trisectors of the angles A and B nearest to AB. Prove that the triangle XYZ is equilateral by elementary geometry.

Solution by the Proposer.

Case (i) *Internal Trisectors.*

Take any equilateral triangle XYZ. Construct triangles YUZ ZVX, XWY, externally, on YZ, ZX, XY, respectively, such that

$$\angle YUZ = \frac{A}{3}, \quad \angle UZY = \frac{B}{3} + 60^\circ, \quad \angle ZYU = \frac{C}{3} + 60^\circ;$$

$$\angle ZVX = \frac{B}{3}, \quad \angle VXZ = \frac{C}{3} + 60^\circ, \quad \angle XZV = \frac{A}{3} + 60^\circ;$$

$$\angle XWY = \frac{C}{3}, \quad \angle WYX = \frac{A}{3} + 60^\circ, \quad \angle YXW = \frac{B}{3} + 60^\circ.$$

Let Y′ and Z′ be the reflections of Y and Z in XW and XV, respectively. Then V, Y′, Z′, W are collinear; for

$$\angle Z'XY' = \angle ZXZ' + \angle YXY' + 60^\circ - 360^\circ = 60^\circ - \frac{2A}{3}$$

and consequently each of the base angles of the isosceles triangle Z′XY′ is equal to $60^\circ + \frac{A}{3}$.

Thus $$\angle XVW = \frac{B}{3}, \qquad \angle XWV = \frac{C}{3}.$$

Thus the theorem holds for triangle UVW and therefore also for the given triangle ABC which is equiangular to UVW.

Case (ii) *External Trisectors.*

The construction and proof are similar to *Case* (i).

Question 1623.

(HANSRAJ GUPTA) :—Prove that

$$\frac{1!\ 2!\ 3!\ 4!\ \ldots\ (r-1)!\ (nr)!}{n!\ (n+1)!\ (n+2)!\ \ldots\ (n+r-1)!}$$

is an integer for all positive integral values of n and r.

Solution by the Proposer.

Let p be any prime number. Then, since the number of times that p occurs as a factor of q ! is $\sum_{k=1}^{\infty} \left[\frac{q}{p^k}\right]$ where $[x]$ denotes the greatest integer in x, it follows that p occurs as a factor of the numerator

$$\sum_{k=1} \left\{ \left[\frac{1}{p^k}\right] + \left[\frac{2}{p^k}\right] + \ldots + \left[\frac{r-1}{p^k}\right] + \left[\frac{nr}{p^k}\right] \right\}$$

$$\equiv \sum_{k=1} \{ \mu(p^k) \} = S_1 \text{ times.}$$

It occurs as a factor of the denominator

$$\sum_{k=1} \left\{ \left[\frac{n}{p^k}\right] + \left[\frac{n+1}{p^k}\right] + \left[\frac{n+2}{p^k}\right] + \ldots + \left[\frac{n+r+1}{p^k}\right] \right\}$$

$$\equiv \sum_{k=1} \{ \delta(p^k) \} = S_2 \text{ times.}$$

We proceed to prove that $S_1 \geqslant S_2$, for all values of p.

Let m be any number of the form p^k. [This restriction is not however essential.]

Let $\quad n \equiv am + n_1, \quad$ where $\quad 0 \leqslant n_1 < m.$

Then,

$$\mu(m) \equiv \left[\frac{1}{m}\right] + \left[\frac{2}{m}\right] + \left[\frac{3}{m}\right] + \ldots + \left[\frac{r-1}{m}\right] + ar + \left[\frac{n\ r}{m}\right]$$

$$\equiv \gamma(m) + ar + \left[\frac{n_1 r}{m}\right]$$

and

$$\delta(m) \equiv ar + \left[\frac{n_1}{m}\right] + \left[\frac{n_1+1}{m}\right] + \left[\frac{n_1+2}{m}\right] + \ldots + \left[\frac{n_1+r-1}{m}\right].$$

Case 1. $m \geqslant r$, so that $\gamma(m) = 0$.

(i) If $n_1 < m-(r-1)$, then $\mu(m) = ar + \left[\frac{n_1 r}{m}\right] \geqslant ar \geqslant \delta(m)$.

(ii) If $n_1 = m-(r-k)$, $k < r$, then $\delta(m) = ar + k$;

and $\mu(m) = ar + \left[\frac{(m-r+k)r}{m}\right]$,

$$= ar + \left[\frac{km + (m-r)(r-k)}{m}\right],$$

$$= ar + k + \left[\frac{(m-r)(r-k)}{m}\right].$$

$\therefore \quad \mu(m) \geqslant \delta(m)$.

Case 2. $m < r$.

Then $\mu(m) = \left[\frac{1}{m}\right] + \left[\frac{2}{m}\right] + \ldots + \left[\frac{r-1}{m}\right] + \left[\frac{nr}{m}\right]$

$$= \gamma(m) + ar + \left[\frac{rn_1}{m}\right]$$

$= ar + \gamma(m) + \left\{ \geqslant \left[\frac{r}{m}\right] + \ldots + \left[\frac{r+n_1-1}{m}\right] \right\}$, since $m < n_1$.

$$\geqslant ar + \left[\frac{1}{m}\right] + \left[\frac{2}{m}\right] + \left[\frac{3}{m}\right] + \ldots + \left[\frac{r+n_1-1}{m}\right];$$

$$\geqslant ar + \left[\frac{n_1}{m}\right] + \left[\frac{n_1+1}{m}\right] + \ldots + \left[\frac{n_1+r-1}{m}\right].$$

$$\geqslant \delta(m)$$

in each case. Hence $\mu(m) \geqslant \delta(m)$, and $S_1 \geqslant S_2$ which proves the result.

Questions 1632, 1633.

(R. VAIDYANATHASWAMY) :—ABCD is a cyclic quadrangle. If the pedal line of A with respect to BCD is perpendicular to the Euler line (*i.e.,* join of circum-centre and centroid) of BCD, prove that a similar property holds good for each of the vertices.

When this property is possessed by ABCD, prove that the perpendiculars let fall from each vertex on its pedal line with respect to the remaining three concur in a point.

(R. VAIDYANATHASWAMY) :—Prove that

$$\tan\frac{a+b+c-d}{2}+\frac{\cos a+\cos b+\cos c}{\sin a+\sin b+\sin c}=0$$

is a symmetric relation between a, b, c, d. Prove furthur that the two relations,

$$\tan\frac{a+b+c+d+\pi-2e}{2}+\frac{\cos a+\cos b+\cos c+\cos d}{\sin a+\sin b+\sin c+\sin d}=0$$

and the similar equation in which d and e are interchanged, constitute two symmetric conditions on a, b, c, d, e.

Indicate the generalisation to n numbers a, b,.........

Solutions by K. Rangaswami.

(i) The first question is a consequence of the known theorem, *viz.*, If a rectangular hyperbola passes through the vertices of a triangle and its circum-centre, it cuts the circum-circle again at the point diametrically opposite to the Euler point* of the triangle.

Given that the pedal line of D with respect to ABC is perpendicular to the Euler line of ABC, the point D is uniquely fixed on the circum circle Ω of ABC as the point diametrically opposite to the Euler point of ABC. From the theorem above stated, the unique rectangular hyperbola Γ through ABC and O, the centre of Ω, passes also through D and is thus the unique conic in the pencil [ABCD] outpolar to Ω. It is in fact the isogonal transformation of the Euler line of ABC and is called the Jerabek's† hyperbola of the triangle.

---

* The Euler point of a triangle is the point whose pedal line with respect to the triangle is parallel to the Euler line of the triangle.

† *Vide* Casey : ***Analytical Geometry***, p. 420.

We may also note that D is the isogonal conjugate of the point at infinity on the Euler line and that Γ passes through the symmedian point of ABC.

The symmetry of the result follows from the fact that Γ circumscribes each of the four triangles and passes through their circum-centre; so that each of the four points is diametrically opposite to the Euler point of the triangle formed by the other three.

If H_a, H_b, H_c, H_d be the ortho-centres of the triangles BCD, etc., AH_a, etc., passes through the centre K of Γ on the nine point circle of ABC. K is also the point of concurrence of the pedal lines of A with respect to BCD, etc. Hence the perpendiculars from A, B, C, D on their corresponding pedal lines concur at the point T on Γ, the *symmetrique* of O with respect to K.

(ii) A uniform proof of the two questions based on circular co-ordinates can be given as follows :—

Let Ω be a unit circle, with centre O, the points A on which are specified by circular co-ordinates

$$t = x + iy = \cos\theta + i\sin\theta\,;\ \bar{t} = x - iy = \cos\theta - i\sin\theta$$

referred to O. Let the suffix r refer to the point A_r and let θ_r be the angular co-ordinate of A_r measured from a line OX with O as pole. It is seen from a known result* that the pedal line of a point A (angular co-ordinate θ_p) with respect A_1, A_2 ......, A_r makes an angle ϕ with OX given by

$$\phi = \sum_{p=1}^{r} \theta_p/2 - (r-2)\,\theta/2 + (r-)\,\pi/2 \qquad \dots \quad (1)$$

Expressing the condition that in a set of r points the pedal line of A with respect to the remaining $(r-1)$ points should be perpendicular to the direction OK $[K \equiv \sum_1^r t_s, \sum_1^r \bar{t}_1]$ we get the relation

$$\left(\frac{1}{t_1} + \dots + \frac{1}{t_{s-1}} + \frac{1}{t_{s+1}} + \dots \frac{1}{t_r}\right) t_1 t_2 \dots t_{s-1} t_{s+1} \dots t_r$$
$$+ (-)^r t_s^{r-3} (t_1 + \dots t_{s-1} + t_{s+1} + \dots + t_r) = 0. \qquad (2)$$

---

* *J. I. M. S.*, Vol. 19, Part II, 1932, p. 269.

whose equivalent in terms of θ is

$$\tan\frac{\theta_1+\ldots+\theta_{s-1}+\theta_{s+1}+\ldots\theta_r-(r-3)\,\theta_s+(r-4)\pi}{2}$$

$$+\frac{\cos\theta_1+\ldots+\cos\theta_{s-1}+\cos\theta_{s+1}+\ldots+\cos\theta_r}{\sin\theta_1+\ldots+\sin\theta_{s-1}+\sin\theta_{s+1}+\ldots+\sin\theta_r}=0. \quad (3)$$

If, in equations (2) and (3), we keep the indices 1, 2, 3 fixed and interchange the others we get $(r-3)$ equations of the form

$$\sigma_{r-2}-\sigma_{r-3}\,t_s+\ldots+(-)^{r-4}\sigma_2\,t_s^{\,r-4}=0 \quad \ldots \quad (4)$$

where σ_p is the product of the t's taken p at a time) of degree $(r-4)$ in t's; and hence (4) is satisfied by $t_1, t_2, \ldots t_r$. Hence $(r-3)$ corresponding equations (*i.e.*, of type 3) in the θ's constitute $(r-3)$ symmetric conditions on $\theta_1, \theta_2, \ldots \theta_r$, which is the required generalisation.

When $r=4$, (2) takes the form

$$\Sigma\, t_p\, t_q, = 0 \quad (p, q = 1, 2, 3, 4) \quad \ldots \quad (5)$$

whose equivalent in terms of θ is

$$\tan\frac{\theta_1+\theta_2+\theta_3-\theta_4}{2}+\frac{\cos\theta_1+\cos\theta_2+\cos\theta_3}{\sin\theta_1+\sin\theta_2+\sin\theta_3}=0 \quad \ldots \quad (6)$$

which proves the first part of the two questions.

Q. 1632 solved analytically by S. Srinivasan and by the use of isotropic co-ordinates by B. Ramamurti.

RECEIVED FOR REVIEW.

Lessons in Elementary Analysis, by G. S. MAHAJANI, M.A., Ph.D. (Cantab.) [Phillip Bayliss Research Student, St. John's College, Cambridge, 1926; Life Member, D. E. Society, Fergusson College, Poona.] Second Edition, Aryabhushan Press Poona, 1934, pp. xii + 264. Price Rs. 5.

ANNOUNCEMENTS AND NEWS.

Readers are invited to contribute to the value of this section by sending suitable news items of interest.

The Silver Jubilee Commemoration Volume of the Society, the publication of which was delayed for various reasons, has now been issued, as Volume XX of the *Journal*. It contains a full report of the proceedings of the Jubilee Conference at Poona and the texts of the papers contributed thereto Copies may be had from S. Mahadeva Iyer, Esq., Assistant Professor, Presidency College, Madras, at a cost of Rs. 10 each.

The Treasurer of the Society, L. N. Subramaniam, Esq., Christian College, Madras, requests all those who have kindly promised contributions to meet the expenses of printing the Jubilee Volume, to remit the donations early to him.

Dr. S. Chowla, M.A., Ph.D., Reader, Andhra University, Waltair, Mr. K. Venkatachaliengar, M.Sc., Research Student, Indian Institute of Science, Bangalore, and Mr. V. A. Mahalingham, M.A., Lecturer, Loyola College, Madras, have been elected members of the Society.

Dr. S. C. Dhar, Professor of Mathematics, College of Science, Nagpur, has proceeded to England for prosecuting higher studies.

The *Scripta Mathematica*, Vol. I, No. 2, gives the following interesting account of the origin of the INSTITUTE FOR ADVANCED STUDY at Princeton.

In 1930 Abraham Flexner published a book entitled "Universities, American, English, German," containing a critical survey of their contributions to the world's advancement. It attracted the attention of L. Bamburger and Mrs. Felix Fuld, former owners of Bamburger and Co., who asked Dr. Flexner what kind of a centre of learning he would organise if adequate funds were placed at his disposal. After consultation with scholars throughout the world, Dr. Flexner decided to start with a School of Mathematics as the "most fundamental of all the disciplines" and for this five million dollars have been allotted. Scholarsphips and Fellowships are available for suitably qualified students without means to pursue advanced studies. Each Professor decides for himself whether to work with seminars or with each student individually, and "if a professor prefers not to give lectures or other formal instruction but to let them simply work alongside and learn as best as they can by their own efforts, he will have complete freedom to do so."

The Directorship of the School was offered to Prof. G. D. Birkhoff but he decided to remain at Harvard. This position as well as a Professorship of Mathematics and Theoretical Physics has now been accepted by Albert Einstein. Other Professors in the School are Oswald Veblen, and Herman Weyl.

Prof. G. D. Birkhoff touched Bombay on his way from Kyoto to Genoa, and was met on behalf of the Society by Prof. Kosambi and Dr. Vijayaraghavan. Prof. Birkhoff received an invitation from the President of the Indian Mathemetical Societys to extend his stay in India.

The complete works of H. A. Lorentz in nine volumes are being prepared for publication under the editorship of P. Zeeman, P Ehrenfest and A. D. Fokker. The Publisher is Martinus Nigoff, of the Hague.

The American Mathematical Society has recently issued the sixteenth volume of its Colloquium publications, *Algebraic Functions* by G. A. Bliss. Price $ 3.00.

A new Journal devoted exclusively to the Theory of Numbers is to be published under the title *Acta Arithmetica*. The first issue is expected to appear in November. The Editorial Board consists of Drs Walfisz Levelski, and Dickstein.

An international journal, *Compositio Mathematica*, is to be published by Noordhoff of Groningen, the Board of Editors consisting of Bieberbach, Brouwer, Donder, Julia and Wilson.

The University of Moscow publishes a Journal entitled "Abhandlungen au dem Seminar fur Vector und Tensoranalysis samt Anwendungen and Geometrie, Mechanik, and Physik" under the editorship of B. Kagan.

We are glad to announce that *The Scripta Mathematica*, a quarterly journal devoted to Philosophy, History, and expository treatment of Mathematics, is being received in exchange by the Society.

The Mathematics Teacher, official organ of the National Council of Teachers of Mathematics, is now being received in exchange for *The Mathematics Student*

Prof. E. H. Neville, of the University of Reading, has been elected President of the (British) Mathematical Association.

Prof. G. N. Watson has been elected President of the London Mathematical Society; Vice-Presidents : Professors A G. Dixon, G. H. Hardy and G. F. J. Temple.

The University of Madras has decided to confer the degree of Doctor of Science on S. Sivasankaranarayana Pillai, B.A., M.Sc., Lecturer in Mathematics in the Annamalai University. The Thesis submitted for the degree relates to certian order results in the theory of numbers and to numbers analogous to Ramanujan's Highly Composite Numbers

Rao Bahadur P. V. Seshu Aiyar has accepted an invitation from the Annamalai University to deliver a course of six lectures on the Theory of Sets of Points. The lectures commence on the 18th October and are open to all who wish to attend.

The Comstock Prize of 2,500 dollars was presented by the National Academy of Sciences to Dr. P. W. Bridgman, Professor of Mathematics and Philosophy at Harvard, in recognition of his brilliant achievements in advancing our knowledge of the behaviour of matter.

Mr. G P. Rao of the Survey of India, Dehra Dun, died on the 7th July 1933

St. Andrews Colloquium.

Prof. Dhar writes from Edinburgh :

Under the auspices of the Edinburgh Mathematical Society, a Colloquium was held at St. Andrews from the 18th July for ten days. A number of distinguished

mathematicians assembled from all parts of the world; notable amongst them being Profs. De Sitter, E. T. Whittaker, E. A. Milne, Brown, and several others.

Prof. H. W Turnbull, F.R.S., of the University of St. Andrews, was the President. The Secretaries were Dr. I. M. H. Etherington, Dr. G. C. McVittie, and Dr. D. E. Rutherford. The following were the lectures delivered :

1. Prof. E. A. MILNE, F.R.S. (Oxford) on "World-Structure by Kinemetic Methods of Special Relativity" (six lectures).

2. Prof. H. W. TURNBULL, F.R.S. (St. Andrews) on "Pictorial Geometry."

3. Prof. B. M. WILSON (Dundee) on "Ramanujan's Note-books and their place in Modern Mathematics" (two lectures).

4. Mr. W. L. FERRAR (Oxford) on "Some Expansions relating to the Problem of Lattice Points" (two lectures).

Prof. B. M. Wilson spoke about Ramanujan and his work in very high eulogising terms, quoting what Prof. Hardy had written about him in the Proceedings of the London Mathematical Society. He and Prof. Watson have undertaken to edit Ramanujan's work and it is expected it will take fifteen years to edit it. He delivered two lectures on Ramanujan's work. Prof. W. De Sitter gave a discourse on the expanding Universe at the Colloquium. Prof. J. M. Whittaker gave a lecture on "Integral Functions."

There were lectures and discussions on several other topics.

The Indian Academy of Sciences, Bangalore.

The first general meeting of the Indian Academy of Sciences was held in the Institute of Science on the 31st of July 1934, when Sir Mirza M. Ismail, Dewan of Mysore, inaugurated the Academy. Many distinguished scientists from different parts of India and visitors were present. At the business meeting held the same day, Sir C. V. Raman was elected as the President of the Academy. Rules and regulations for the conduct of the Academy were passed. An influential council was appointed with Rao Bahadur Prof. B. Venkatesachar and Prof. C. R. Narayana Rao as Secretaries.

The first number of the Proceedings of the Academy was issued to the Fellows the same day.

The 1st and 2nd of August were spent in the discussion of many papers presented to the Academy. A discussion on molecular spectra was led on the 3rd by Prof. Samuel of the Aligarh University and a few papers of interest on the same subject were read on the 4th of August.

Business meetings of the Academy will be held once every month when papers contributed will be read and discussed. An annual meeting will be held in July every year.

ACTIVITIES OF STUDENTS' ASSOCIATIONS.

University of Dacca.

Dr. M. Basu writes :—The University Mathematics Seminar meets once a week and Honours and post-graduate students are encouraged to speak. Last session the following First Year Honours students spoke on the subjects mentioned.

M. C. Chakravarty :	"On Transcendental Numbers."
Nawab Ali Meah :	"Amsler's Planimeter."
Manindra Kumar Roy :	"Legendre's Polynomials."
A. K. Paul :	"On Gamma Functions."

Loyola College Mathematics Association.

Leecturer.	Subject.
Dr. R. Vaidyanathaswamy, D.Sc. ...	Time and the Theory of Relativity
H. H. Jagadguru Sri Sankaracharya of Puri Govardhan Mutt	Mathematics and other Positive Sciences, Ancient and Modern.
Rao Bahadur P. V. Seshu Ayyar, B.A., L.T.	The Nature of Mathematics and its relation to other Sciences.
Mr. M. Narayanamurti, M.A. ...	Wireless.
Mr. P. V. Natesan, M.A. ...	The Theory of the Simple Voltaic Cell.
Mr. M. Narayanamurti, M.A. ...	Wireless.
Dr. H. Parameswaran, M.A., D.Sc. ...	Technical Education in India.
Mr. A. Somayajulu, M.A. ...	Some Aspects of Hindu Mathematics.

Annamalai University Mathematics Association.

The following papers were read during the academic year 1933-34 :—

P. Jagannadhan	...Class V Hons....	"Waring's Probem."
V. T. Srinivasan	... ,, ...	"Transfinite Numbers."
N. Ramachandran	... ,, ...	"Transcendence of e."
A. N. Sankaran	... ,, ...	"Calculus of Finite Differences"
S. Ramaswamy	... ,, ...	"Vectorial mechanics."
R. Subrahmanyam	... ,, ...	"Number, the Language of Mathematics."
V. Seetharaman	... ,, ...	"The curve of quickest descent under gravity."
S. Pichai	... ,, ...	"Theory of sets of points."
K. Rangaswamy, B.A.	Hons. Research Student. ...	"The infinite Regions of various geometries."

QUESTIONS FOR SOLUTION.

Proposers of Questions are requested to send their own Solutions along with their Questions.

1677. (ALFRED MOESSNER) :—We have

$$7^n + 8^n = 1^n + 5^n + 9^n \qquad (n = 1, 3).$$

Is the equation $M_1^n + M_2^n = N_1^n + N_2^n + N_3^n \qquad (n = 1, 3)$ also solvable when $M_1 = M_2$?

1678. (HANSRAJ GUPTA) :—If $|x| < 1$, then

$$\frac{1}{(1-x^2)(1-x^3)(1-x^4)\ldots} = 1 + \sum_{s=1}^{\infty} \frac{x^{2s}}{(1-x)(1-x^2)\ldots(1-x^s)}$$

1679. (RATAN LAL GUPTA) :—Show that

$$\text{(i)} \quad \int_0^y \frac{x \sin ax}{\sqrt{y^2 - x^2}}\, dx = \frac{\pi}{2}\, y\, J_1(ay)$$

$$\text{(ii)} \quad \int_0^{\pi/2} J_1(y \sin \theta)\, d\theta = \frac{1 - \cos y}{y}$$

where J_1 is the Bessel function of the first order.

1680. (K. SATYANARAYANA) :—A pair of real isogonal conjugates with respect to a given triangle exists on a given line when and only when the line cuts a pair of opposite sides of the ortho-centric tetrad (formed by the in- and ex- centres) one internally and one externally.

1681. (A. A. KRISHNASWAMI AYYANGAR) :—If the F-conic of two circles degenerates, show that it is either a pair of parallel straight lines or a pair of perpendicular straight lines; and in the latter case, show that the perpendicular lines meet at one of the limiting points of the co-axal system determined by the circles, the other limiting point being the centre of the degenerate ϕ-conic.

1682. (K. RANGASWAMI) :—The feet of the normals from a point P to a given conic S lie on a rectangular hyperbola H through P. Show that if H passes through a fixed point Q in the plane, the locus of P is a straight line l through Q.

Show also that if Q moves along a diameter of S, l remains parallel to itself.

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THE MATHEMATICAL ASSOCIATION OF AMERICA

The Mathematical Association of America is devoted primarily to the interests of collegiate mathematics. It now has over two thousand individual and institutional members. There are twenty sections at present, representing thirty-one different States. The Association holds two national meetings per year and the sections hold one or two meetings per year. All meetings, both national and sectional, are reported in the Official Journal.

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All communications intended for publication should be sent to the Editor, A. NARASINGA RAO, Annamalai University, Annamalainagar, South India. Communications relating to the receipt of the Journal, changes of address, and the purchase of current or back numbers of this journal should be sent to the Assistant Secretary, S. MAHADEVA IYER, Assistant Professor of Mathematics, Presidency College, Madras.

Remittances of membership fees and of subscriptions should be sent to the Treasurer, Prof. L. N. SUBRAMANIAM, Department of Mathematics, Christian College, Madras.